Notes on the modeling of irregular seas in time simulations

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R1	2010-10-19	Distributed to members of the Ship Dynamics research program at KTH.
R2	2010-11-29	Error in eq.(5.1) corrected. Small editorial corrections.

Executive summary

Time series simulation of ships dynamic behaviour in waves needs to be based on a wave sequence that represents irregular sea states in a realistic way. This short note discusses the quality of wave sequence generation based on standard sea spectra and has been set up to document some experience gained as a side product of other studies.

It is recommended to use a transformation of the traditional frequency based wave spectrum to a period based spectrum for the purpose of simulation of wave sequences and wave induced effects on ships such as motions and dynamic stability variation. This transformation will enable the use of fewer components and at the same time a very high statistical quality and unlimited return periods before the wave pattern will be repeated.

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Notes on the modeling of irregular seas in time simulations

1 Introduction

Time series simulation of ships dynamic behaviour in waves needs to be based on a wave sequence that represents irregular sea states in a realistic way. This short note has been set up to document some experience gained as a side product of other studies. It discusses the quality of wave sequence generation based on standard sea spectra and gives some recommendations on suitable modelling approaches.

2 Standard wave spectrum formulation in the frequency domain

The standard form of a wave energy spectrum illustrates the distribution of energy of the frequency domain. It can be seen as a Fourier transform of measuring sample of the water surface elevation where the irregular wave pattern is broken down to harmonic components.

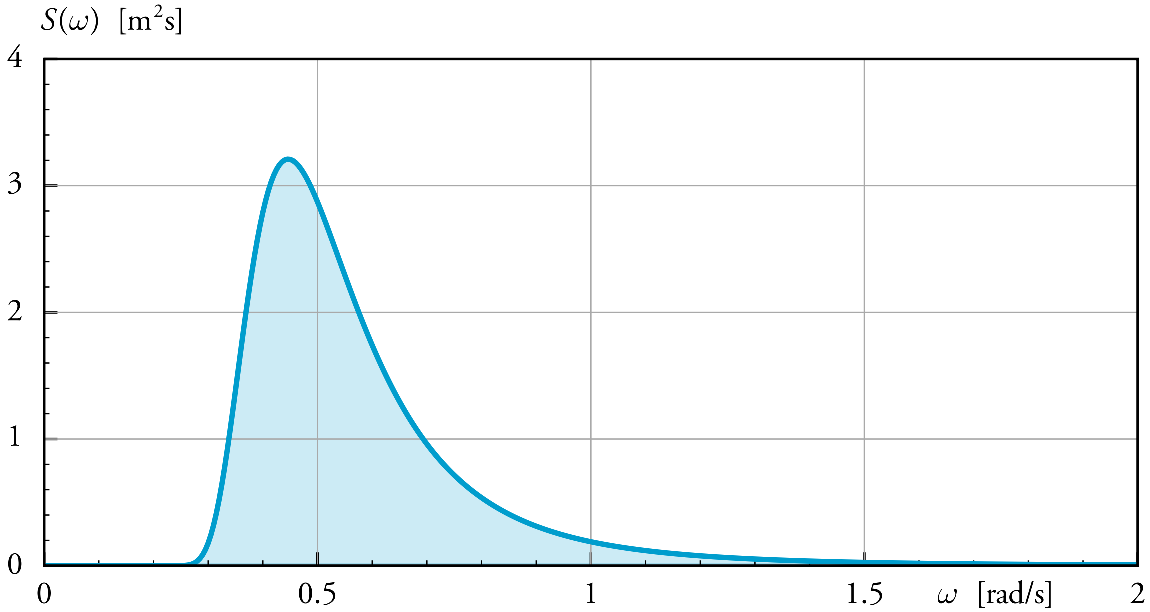


Figure 2.1 Example of wave energy spectrum in the frequency domain
(standard 2-parameter Bretschneider spectrum H_s=4m, T_z=10s)

The shape of the wave spectrum reflects the character of the irregular sea. A spectrum with large area represents a severe sea state with large waves, a spectrum that is spread out over a wide span of frequencies represent a very chaotic sea state with a mixture of short and long waves while a narrow spectrum represents a rather regular sea state (such as swell) where most of the energy is concentrated to waves with almost the same frequency.

The water surface elevation at a certain position and time is a stochastic variable in a real (random) irregular sea. The spectrum can therefore never give the exact deterministic description of the sea but only statistical parameters that describe the character of the sea, and vice versa can standard spectra be defined if these statistical parameters are measured.

The characteristic parameters are usually defined by the moments of the spectrum

$$m_n = \int \omega^n S(\omega)\, d\omega \tag{2.1}$$

The most commonly used description of wave heights is the significant wave height H_s or $H_{1/3}$ which is defined as the average of the 1/3 largest wave heights and approximately corresponds to the visual appearance of typical wave heights in irregular seas (the human eye filters out the lower components).

$$H_s = 4\sqrt{m_0} \quad \text{(for the spectrum shown in Figure 2.1, } H_s = 4\,\text{m)} \tag{2.2}$$

For statistical characteristic values of wave periods unfortunately there are several different definitions dependent on the context. The following list contains the most used alternatives but the denominations are not standard.

$$T_p = \frac{2\pi}{\omega_m} \quad \text{peak period where } \omega_m \text{ is the modal frequency corresponding to the maximum energy} \tag{2.3}$$

(in Figure 2.1, ω_m is 0.45 rad/s and T_p is 14.0 s)

$$T_{-1} = 2\pi \frac{m_{-1}}{m_0} \quad \text{mean (meteorological) period (in Figure 2.1, } T_{-1} \text{ is 12.0 s)} \tag{2.4}$$

$$T_1 = 2\pi \frac{m_0}{m_1} \quad \text{mean (average) period (in Figure 2.1, } T_1 \text{ is 10.8 s)} \tag{2.5}$$

$$T_z = T_0 = T_2 = 2\pi \sqrt{\frac{m_0}{m_2}} \quad \text{mean (zero crossing) period (in Figure 2.1, } T_z \text{ is 10.0 s)} \tag{2.6}$$

$$T_{max} = 2\pi \sqrt{\frac{m_2}{m_4}} \quad \text{mean period between maxima (in Figure 2.1, } T_{max} \text{ is 6.2 s)} \tag{2.7}$$

In similar way one can define a statistical wave length

$$\lambda_0 = g \frac{T_z T_{max}}{2\pi} \quad \text{mean apparent wave length (in Figure 2.1, } \lambda_0 \text{ is 97 m)} \tag{2.8}$$

The most simple standard wave spectra use a fixed shape formulation that is tuned by only two parameters, significant wave height and any of the characteristic periods listed above (2-parameter Bretschneider spectra). Figure 2.1 is based on the following formulation

$$S(H_s, T_z, \omega) = \frac{H_s^2 T_z}{8\pi^2} \left(\frac{2\pi}{\omega T_z}\right)^5 e^{-\frac{1}{\pi}\left(\frac{2\pi}{\omega T_z}\right)^4} \tag{2.9}$$

However, it must be stressed that real world wave spectra seldom are as smooth as such standardised spectra. There may for instance well be several peaks for different dominating frequencies.

3 Wave spectrum formulation in the period domain

An alternative and (in my view) more physical way to describe the irregular sea is to transform the standard wave frequency domain to the wave period domain. The condition to be satisfied for such a transformation is that the total energy, described by the integral of the spectrum, is to be maintained.

$$\int_{\omega_{\min}}^{\omega_{\max}} S(\omega)\,d\omega = \int_{T_{\min}}^{T_{\max}} S(T)\,dT$$

with $\omega = \dfrac{2\pi}{T}$ we get

$$\int_{\omega_{\min}}^{\omega_{\max}} S_\omega(\omega)\,d\omega = \int_{T_{\max}=\frac{2\pi}{\omega_{\min}}}^{T_{\min}=\frac{2\pi}{\omega_{\max}}} S_\omega(\omega)\frac{d\omega}{dT}dT = \int_{T_{\max}}^{T_{\min}} S_\omega(\frac{2\pi}{T})\frac{-2\pi}{T^2}dT = \int_{T_{\min}}^{T_{\max}} S_\omega(\frac{2\pi}{T})\frac{2\pi}{T^2}dT = \int_{T_{\min}}^{T_{\max}} S_T(T)\,dT$$

where

$$S_T(T) = S_\omega(\frac{2\pi}{T})\frac{2\pi}{T^2} = S_\omega(\omega)\frac{\omega^2}{2\pi} \tag{3.1}$$

The standard 2-parameter spectrum in equation (2.9) can thus alternatively be written

$$S(H_s,T_z,T) = \frac{H_s^2 T_z}{4\pi T^2}\left(\frac{T}{T_z}\right)^5 e^{-\frac{1}{\pi}\left(\frac{T}{T_z}\right)^4} \tag{3.2}$$

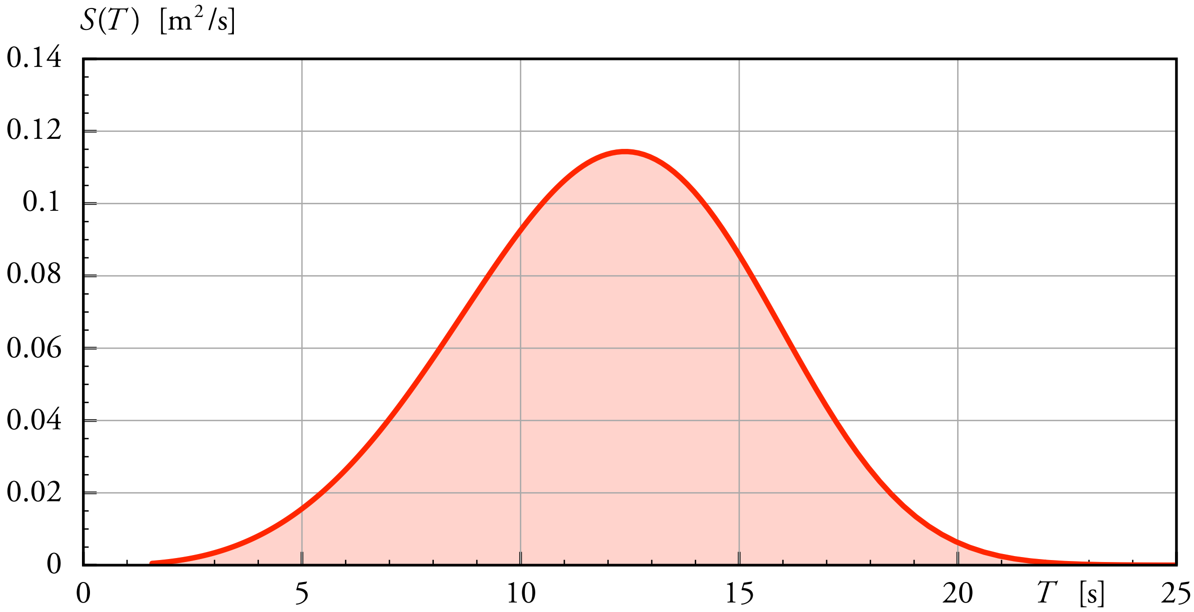


Figure 3.1 Example of wave energy spectrum in the period domain
(same standard 2-parameter Bretschneider spectrum as in Figure 2.1, H_s=4m, T_z=10s)

This transformation has several advantages compared to the standard formulation in the frequency domain. Firstly, it is possible to interpret intuitively (I can't imagine anyone describing seas in terms of angular frequencies…); a period-based spectrum will be much easier to communicate with the officers on board. Secondly, as will be shown in the following sections, a period based spectrum is much better suited for discretisation and is superior as basis for numerical simulations of irregular seas.

In the frequency domain, the major part of the energy is concentrated to a narrow band of frequencies while the rest of the energy is spread out over a relatively long tail with higher frequencies. Transformed to the period domain, we can clearly see that the energy distribution is much better represented with little skew and a well defined range of periods without long tails.

The mean (meteorological) period T_{-1} defined in (2.4) is well representing both the average and the peak of the transformed spectrum while the "peak" period T_p calculated from the modal frequency (2.3) shows off not to correspond to the period with the highest energy!

It should be stressed that although these two representations of the wave energy distribution are identical the transformation makes it clear that from a semantic point of view that one should avoid expressing characteristic periods from the frequency spectrum (and vice versa).

4 Wave sequence generation for numerical simulations

When generating sample sequences of waves in numerical simulations, the standard solution is to divide the frequency spectrum into a number of discrete harmonic wave components (unfortunately often with equally spaced frequencies) that are superposed with random phase lag.

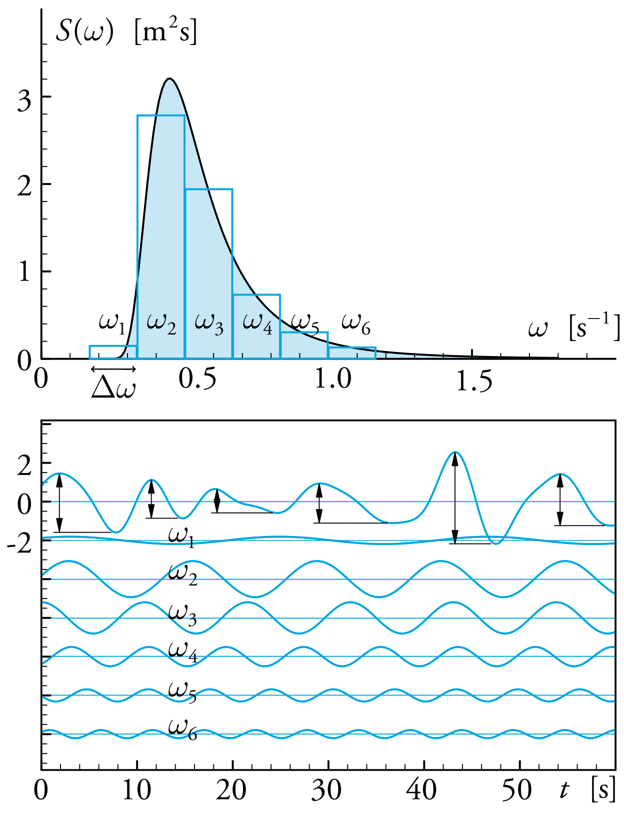


Figure 4.1 A simplified illustration of how an irregular sequence of waves can be simulated from a wave spectrum. The quasi-random wave height is indicated by the arrows.

The amplitude of each harmonic component in the simulation is calculated from the spectrum as

$$a_i = \sqrt{2 \int_{\omega_i - 0.5\Delta\omega}^{\omega_i + 0.5\Delta\omega} S(\omega)\, d\omega} \tag{4.1}$$

There are two major quality indicators of the sampling:

1. The ability to produce long (or infinite) return periods T_r i.e. the total time span that a certain set of random phase lags may generate a unique sequence of waves that is not repeated.
2. The ability to produce a (theroretically) correct distribution of time samples:
In an irregular sea composed of infinite number of wave components, the surface elevation shall according to the central limit theorem by normal (Gauss-) distributed with the variance equal to the area under the spectrum m_0. The distribution of amplitudes should follow close to a Rayleigh distribution (dependent on the bandwidth) and the simulation shall be able to produce large amplitudes during long sequences of sampling.

Although these indicators are somewhat linked, they can be adjusted by different means. Let us look at each of them separately and compare the two alternative spectrum formulations.

4.1 The return period of sampled sequences

The condition for a repeated sequence is that all harmonic components of the superposed wave profile must have run through an (integer) number of full periods at the same time. This can be used to form a set of conditions for the return period T_r. For all components the following condition must be satisfied:

$$\left(\frac{\omega_1}{2\pi} + (i-1)\frac{\Delta\omega}{2\pi}\right) T_r = k_i$$

where

ω_1 is the frequency of the first component

i is the component number, $i = 1, 2 \ldots, n$ where n is the total number of components (4.2)

$\Delta\omega$ is the constant step in frequency between components

T_r is the return period for the superposed wave pattern

k_i is an integer number

Since

$k_1 = \frac{\omega_1}{2\pi} T_r$ is integer,

then (4.2) brakes down to satisfying the following condition

$$k_2 = k_1\left(\frac{\Delta\omega}{\omega_1} + 1\right), \quad [\omega_1 > 0] \quad \text{or} \quad k_2 = 1, \quad T_r = \frac{2\pi}{\Delta\omega}, \quad [\omega_1 = 0] \tag{4.3}$$

which then will satify all other conditions for $i = 3, 4 \ldots n$ because

$k_1 + (i-1)(k_2 - k_1) = k_i$ must by default always be integer if k_1 and k_2 are integer

It is worth noting that the conditions are in principle independent of n, we can get a very short return period even if the spectrum is divided into hundreds of harmonic components!

Lets take two examples from the same spectrum as illustrated in the previous figures.

The first example:

$n = 10$

$\omega_1 = 0.2 \quad \Delta\omega = 0.22$ which gives $\omega_i = 0.2, 0.42, \ldots, 2.18$ [rad/s]

then we get $k_1 = 10$ and $k_2 = 21$ satisfying (4.3), which gives

$$T_r = \frac{10 \cdot 2\pi}{\omega_1} = 100\pi = 314.2\,\text{s}$$

The second example:

$n = 100$

$\omega_1 = 0.2 \quad \Delta\omega = 0.02$ which gives $\omega_i = 0.20, 0.22, \ldots, 2.18$ [rad/s]

then we get $k_1 = 10$ and $k_2 = 11$ satisfying (4.3), which gives exactly the same return period!

$$T_r = \frac{10 \cdot 2\pi}{\omega_1} = 314.2\,\text{s}$$

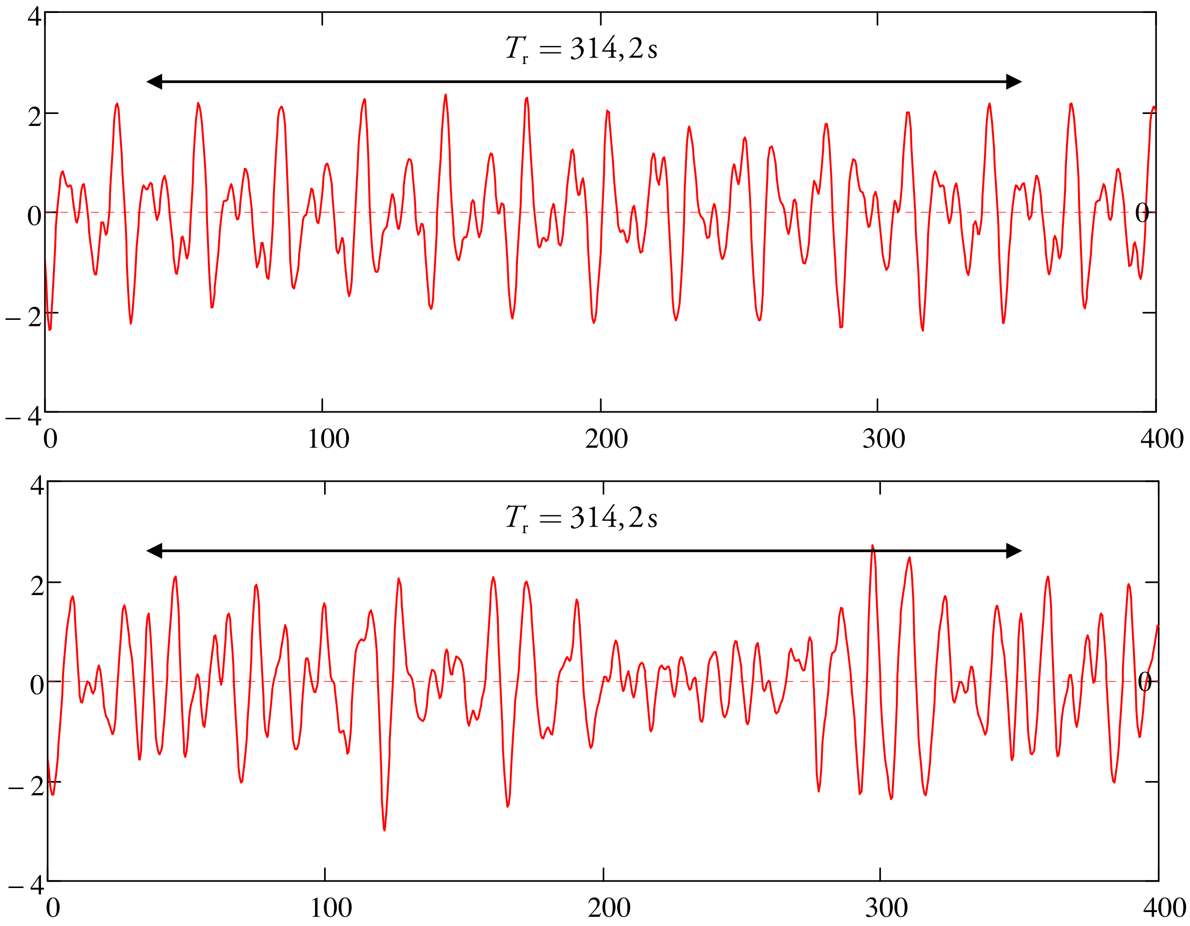


Figure 4.2 Two examples of simulated wave sequences n=10 (top) and n=100 (bottom) with the same return period. Note the more regular pattern in the first example.

One way of extending the return period for the superposed frequency components is to make the ratio $\frac{\Delta\omega}{\omega_1}$ as far from a simple rational expression as possible (or even non-rational e.g. by expressing one of the two frequencies as a fraction of e). However, even if the theoretical return period thus becomes infinite, the general character will still be very close to repeated shorter sequences.

If we instead use the period based spectrum with constant steps in the period domain, the return period must satisfy:

$$\frac{T_r}{T_1 + (i-1)\Delta T} = k_i \qquad (4.4)$$

where

T_1 is the period of the first component

i is the component number, $i = 1, 2 \ldots, n$ where n is the total number of components

ΔT is the constant step in periods between components

T_r is the return period for the superposed wave pattern

k_i is an integer number

This condition gives much longer return periods because we have to find the common denominator of all components. It can be shown that the return period will be

$$T_r \geq T_1 + \Delta T(n-1)! \qquad (4.5)$$

If we are to cover a range of periods from 1s-25s with 10 components with equal steps in periods, then

$$T_1 = 1, \quad \Delta T = \frac{25-1}{9}, \quad T_r = 1 + \frac{25-1}{9}(9)! = 1 + 24 \cdot (8)! = 967681\,\text{s} \approx 11.2\,\text{days}$$

We see that the return period will never be a problem with a reasonable number of components if we use constant steps in the period domain.

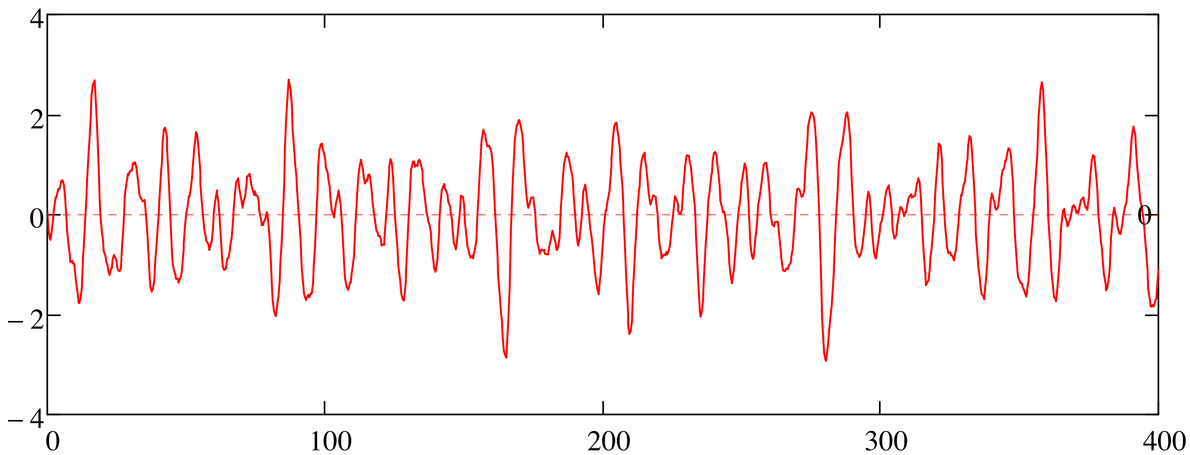


Figure 4.3 Example of a simulated wave sequence from a period spectrum divided in 10 components. The irregular character and quality is similar to the bottom sequence in Figure 4.2 (n=100), the return period is more than 2400 times the shown period.

4.2 Statistical quality of sampled wave sequence

In the following, we shall compare discretisation with equal steps in the frequency domain and period domain respectively. For a single harmonic surface wave, the distribution of momentary measures of the water level will follow a frequency distribution defined by[1]

$$f(\zeta) = \frac{1}{\pi\sqrt{a^2 - \zeta^2}} \tag{4.6}$$

With a superposition of a large number of wave components with random phase the water level will according to the central limit theorem approach the normal distribution

$$f(\zeta) = \frac{1}{\sqrt{2\pi\, m_0}} e^{-\frac{\zeta^2}{2m_0}} \tag{4.6}$$

where m_0 that is the wave spectrum area according to (2.1) will be the variance of the normal distribution.

In order to illustrate the difference between the two spectrum formulations, a 3600s wave sequence have been simulated with different number of components n=5, 10, 50 and 100 from the same spectra as used in previous examples. In the frequency domain the components covers the range from 0.2 to 2.18rad/s and in the period domain the range of periods is 1.0 to 25s. The water level has been sampled with a frequency of 5Hz and the distribution is illustrated with histograms in Figure 4.4. In the figure also the theoretical sample maximum amplitude (i.e. the sum of all component amplitudes) and the actual measured max amplitudes and standard deviation during on hour is given. The standard deviation shall theoretically be 1.00 (and assuming a zero bandwidth the theoretical most probable maximum amplitude should be approximately ±3.4m).

It should be emphasised that the comparison is just one sample out of an infinite number of possible random sequences. However, the result is typical and indicative. From any point of view, the wave simulation from a period spectrum will be superior to that from a frequency spectrum. Even with as low as 10 components, the quality of the simulation is very good.

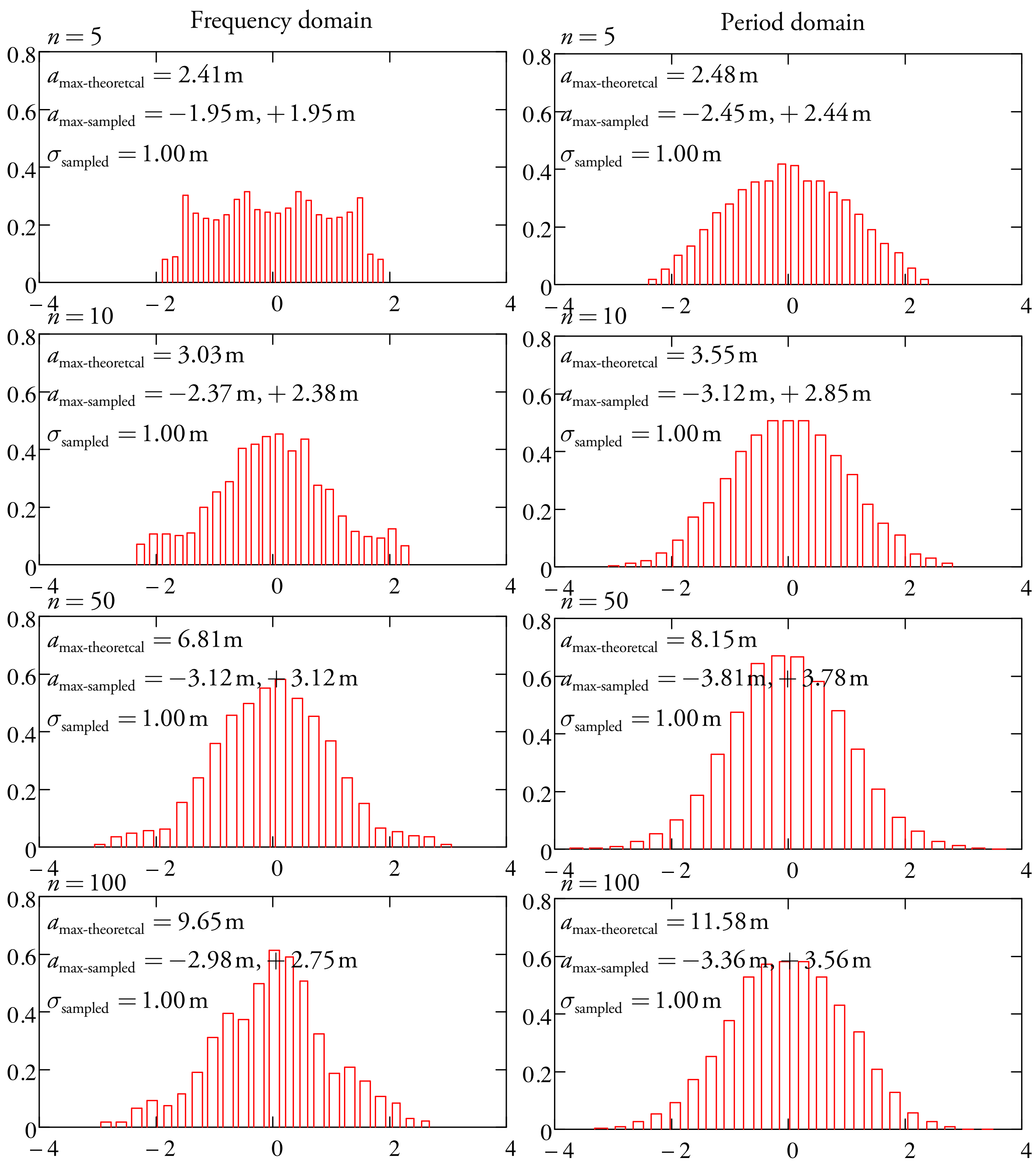


Figure 4.4 Comparison of 5Hz samples from one hour simulation of wave pattern in an irregular sea with significant wave height H_s=4m and zero crossing period T_z=10s. The random phase lags between components have been kept constant throughout the simulation. (The histogram y-axis is normalized so the level should be disregarded)

5 A possible new standard wave simulation model

It would be convenient to try to establish one standard for wave simulations that would cover all kind of wave spectra. This would facilitate simulation code development and enable better benchmarking opportunities.

The 2-parameter Bretschneider spectrum used here to illustrate the advantages with a period based discretisation, was developed to describe fully developed seas on the deep oceans and has a large bandwidth compared to spectra in more restricted areas (such as the JONSWAP spectra). However in real seas there is often a mixture of new and old sea states that may create very broad spectra with several local peaks. In order to be able to capture the specific characteristics of such conditions, the simulation model has to be sufficiently broad and sufficiently detailed.

Figure 5.1 below shows the span of periods covered by 2-parameter spectra with zero crossing periods from 3s to 15s. In order to have one single simulation scheme to cover all this range with high quality, it is suggested that the period increment should be about 0.5s starting from 0.5s and the upper limit above 2.3*T_z.

An possible alternative would be to use a fixed number of components over the interesting range, from 0.5s to 2.3*T_z. From a quality point of view it would be fully sufficient to use about 20 components to cover the range of interest. The drawback of this alternative is of course that the discrete component periods will not be the same for different wave spectra.

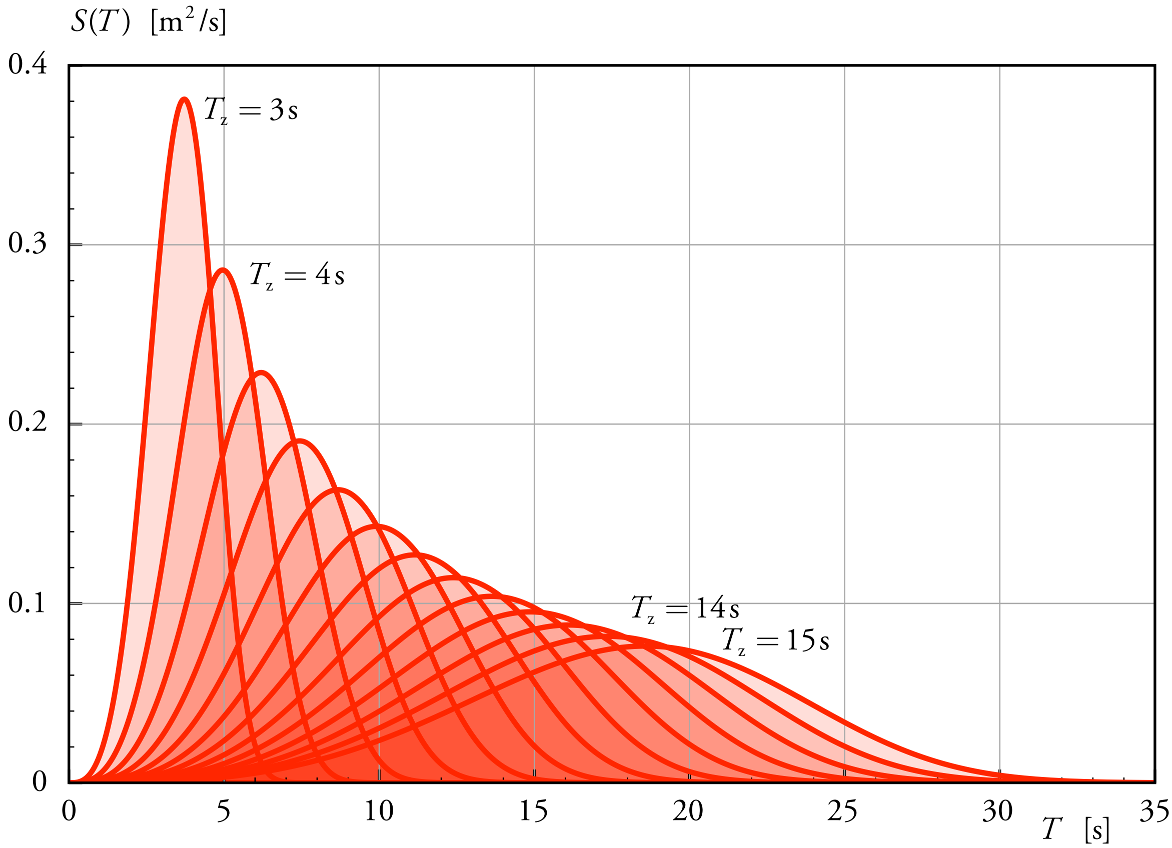


Figure 5.1 Wave spectra in the period domain for various zero crossing periods.
The significant wave height is set to constant 4m for comparison but may not be realistic for the very short periods.

A proposed standard simulation model with fixed periods increments of 0.5s can thus be formulated as

$$\zeta(t) = \sum_{i=1}^{0.5i>2.3T_z} \left[\sqrt{2 \int_{0.5i-0.25}^{0.5i+0.25} S(T)\,dT} \cdot \cos\left(\frac{2\pi}{0.5i} t + \varepsilon_i \right) \right] \tag{5.1}$$

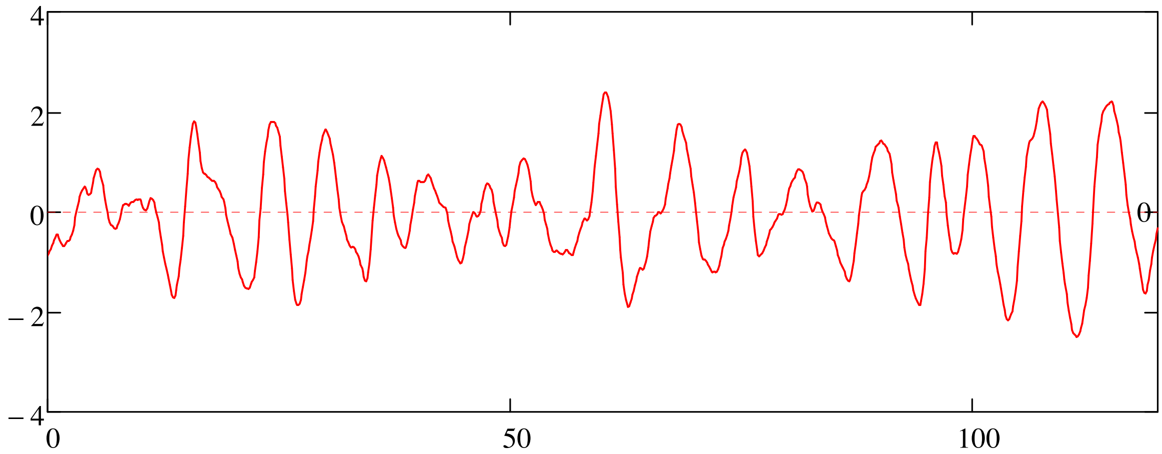
where $S(T)$ is the transformed wave spectra according to (3.1).

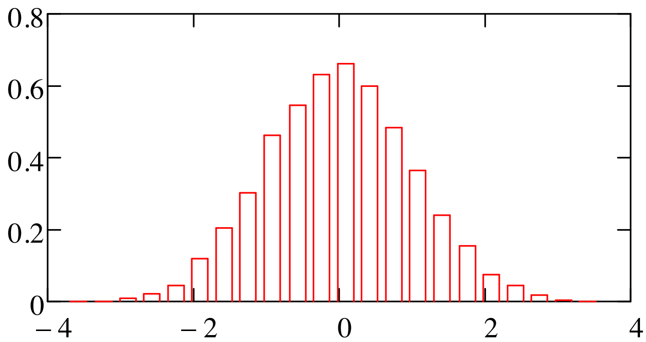
The alternative formulation using a fixed number of n (20 is sufficient) period components becomes

$$\zeta(t) = \sum_{i=1}^{n} \left[\sqrt{2 \int_{0.5+\frac{(i-1.5)}{(n-1)}2.3T_z}^{0.5+\frac{(i-0.5)}{(n-1)}2.3T_z} S(T)\,dT} \cdot \cos\left(\frac{2\pi}{0.5 + \frac{(i-1)}{(n-1)} 2.3T_z} t + \varepsilon_i \right) \right] \tag{5.2}$$

Both will yield excellent wave sequences with, in practice unlimited return periods for any foreseeable irregular wave condition of interest for ship analysis. The first (5.1) could be preferable when used with pre-calculated transfer functions, while the second (5.2) may be more efficient in a direct simulation.

Figures 5.2-5.4 shows examples using (5.1) and (5.2) for three different zero crossing periods 6s, 9s and 12s. The time sequence shown is $10T_z$ but the statistical histogram is based on one hour sampling.



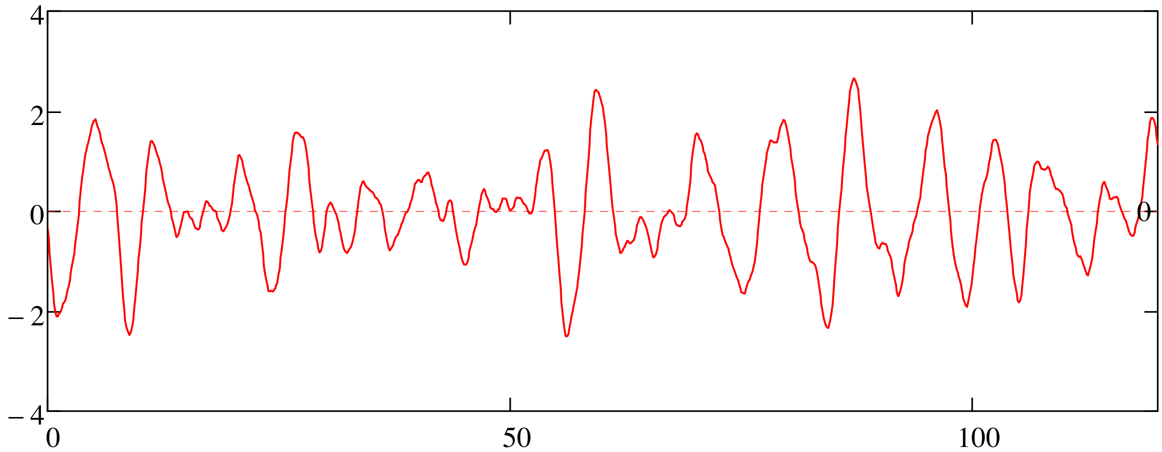


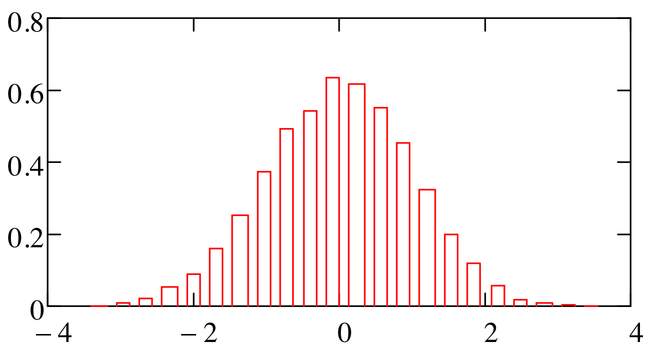
$H_s = 4 \qquad T_z = 6 \qquad n = 28$

$a_{max\_theoretical} = 6.247$

$a_{max\_sampled} = (-3.739 \quad 3.577)$

$\sigma_{sampled} = 0.999$





$H_s = 4 \qquad T_z = 6 \qquad n = 20$

$a_{max\_theoretical} = 5.286$

$a_{max\_sampled} = (-3.454 \quad 3.619)$

$\sigma_{sampled} = 0.999$

Figure 5.2 Simulation of 2-parameter spectrum with H_s=4m and zero crossing period T_z=6s. The example at top is based on (5.1) and at bottom on (5.2)

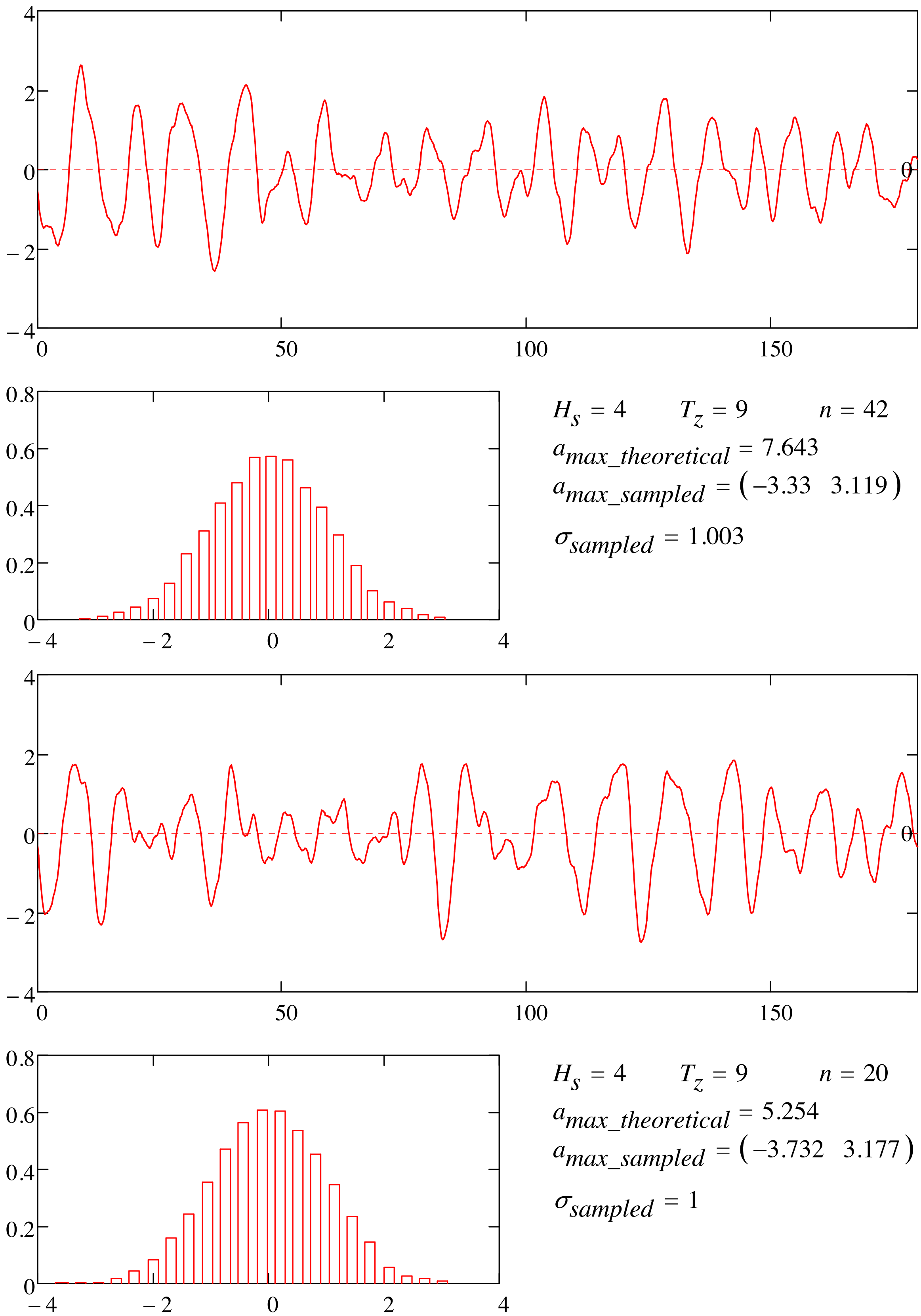


Figure 5.3 Simulation of 2-parameter spectrum with H_s=4m and zero crossing period T_z=9s. The example at top is based on (5.1) and at bottom on (5.2)

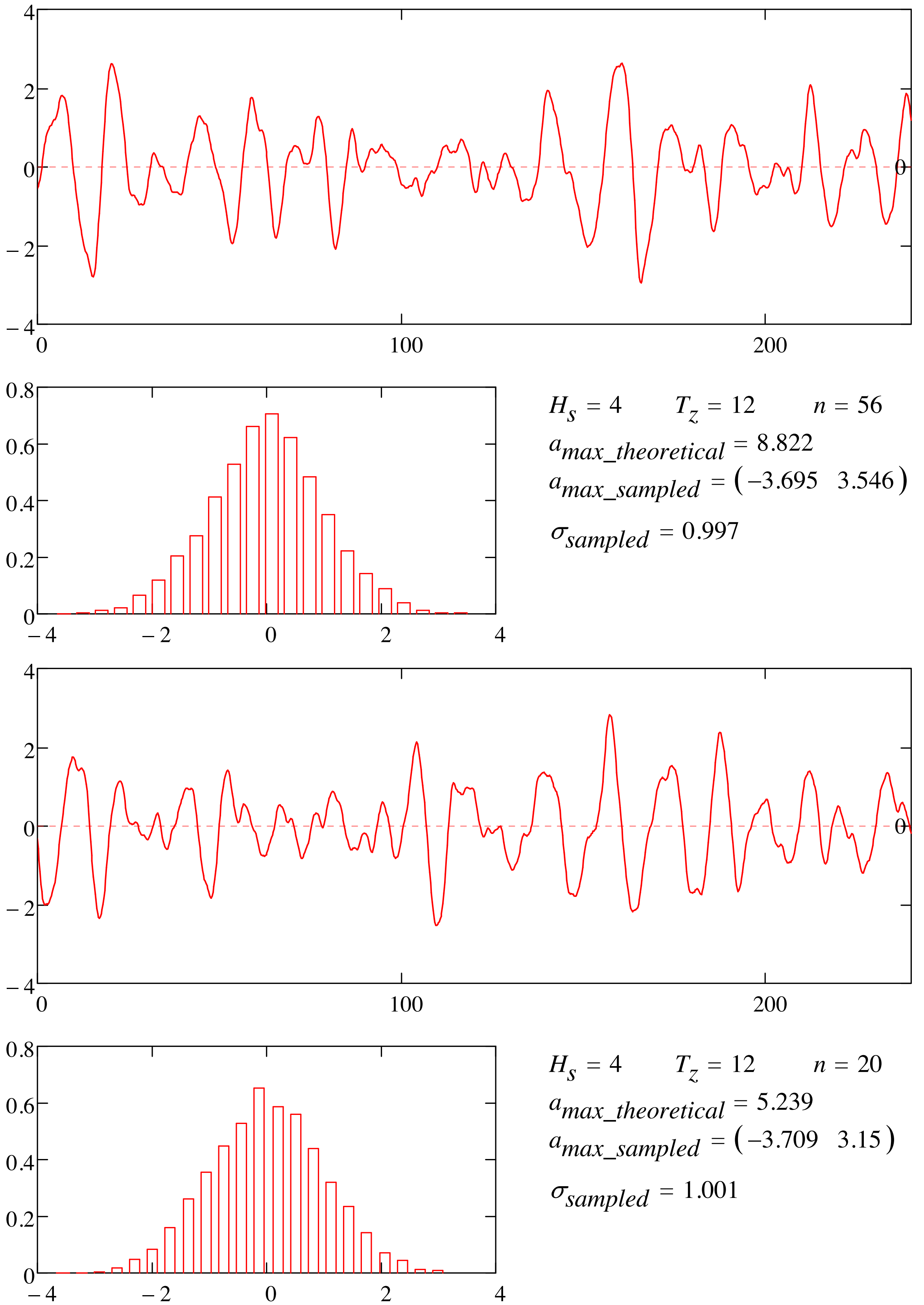


Figure 5.4 Simulation of 2-parameter spectrum with H_s=4m and zero crossing period T_z=12s. The example at top is based on (5.1) and at bottom on (5.2)

6 Recommendations

It is highly recommended to use a transformation of the traditional frequency based wave spectrum to a period based spectrum for the purpose of simulation of wave sequences and wave induced effects on ships such as motions and dynamic stability variation. This transformation will enable the use of fewer components and at the same time a very high statistical quality and unlimited return periods before the wave pattern will be repeated.

For benchmarking purposes it is further recommended to standardise the wave modelling so that actual simulation sequences can be repeated. With fixed period increments and an unlimited return period the random phase lag between components could be locked so that specific wave patterns can be identified by referring just to a specific sequence in time.

7 References

1 *Statistiska metoder för beräkning av Vågor och Gensvar*,
lecture notes, M Huss 1983 (www.mhuss.se/documents/Downloads/Vagor&Gensvar.pdf)